Example 33 Coloured balls are distributed in four boxes as shown in the following table:

Box	**Colour**			
	Black	**White**	**Red**	**Blue**
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III?

Solution Let A, E_1, E_2, E_3 and E_4 be the events as defined below :

A : a black ball is selected E_1 : box I is selected

E_2 : box II is selected E_3 : box III is selected

E_4 : box IV is selected

Since the boxes are chosen at random,

Therefore $$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$$

Also $$P(A|E_1) = \frac{3}{18},\ P(A|E_2) = \frac{2}{8},\ P(A|E_3) = \frac{1}{7} \text{ and } P(A|E_4) = \frac{4}{13}$$

P(box III is selected, given that the drawn ball is black) = $P(E_3|A)$. By Bayes' theorem,

$$P(E_3|A) = \frac{P(E_3)\cdot P(A|E_3)}{P(E_1)P(A|E_1)+P(E_2)P(A|E_2)+P(E_3)P(A|E_3)+P(E_4)P(A|E_4)}$$

$$= \frac{\frac{1}{4}\times\frac{1}{7}}{\frac{1}{4}\times\frac{3}{18}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{7}+\frac{1}{4}\times\frac{4}{13}} = 0.165$$

Example 34 Find the mean of the Binomial distribution $B\left(4, \frac{1}{3}\right)$.

Solution Let X be the random variable whose probability distribution is $B\left(4, \frac{1}{3}\right)$.

Here $$n = 4,\ p = \frac{1}{3} \text{ and } q = 1 - \frac{1}{3} = \frac{2}{3}$$

We know that $$P(X = x) = {}^4C_x\left(\frac{2}{3}\right)^{4-x}\left(\frac{1}{3}\right)^x,\ x = 0, 1, 2, 3, 4.$$

i.e. the distribution of X is

x_i	$P(x_i)$	$x_i\,P(x_i)$
0	${}^4C_0\left(\frac{2}{3}\right)^4$	0
1	${}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$	${}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)$

2	${}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$	$2\left({}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2\right)$
3	${}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3$	$3\left({}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3\right)$
4	${}^4C_4\left(\frac{1}{3}\right)^4$	$4\left({}^4C_4\left(\frac{1}{3}\right)^4\right)$

Now Mean $(\mu) = \sum_{i=1}^{4} x_i\, p(x_i)$

$$= 0 + {}^4C_1\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 2\cdot{}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 3\cdot{}^4C_3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + 4\cdot{}^4C_4\left(\frac{1}{3}\right)^4$$

$$= 4\times\frac{2^3}{3^4} + 2\times6\times\frac{2^2}{3^4} + 3\times4\times\frac{2}{3^4} + 4\times1\times\frac{1}{3^4}$$

$$= \frac{32+48+24+4}{3^4} = \frac{108}{81} = \frac{4}{3}$$

Example 35 The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Solution Let the shooter fire n times. Obviously, n fires are n Bernoulli trials. In each trial, p = probability of hitting the target = $\frac{3}{4}$ and q = probability of not hitting the target = $\frac{1}{4}$. Then $P(X = x) = {}^nC_x\, q^{n-x} p^x = {}^nC_x\left(\frac{1}{4}\right)^{n-x}\left(\frac{3}{4}\right)^x = {}^nC_x\frac{3^x}{4^n}$.

Now, given that,

P(hitting the target at least once) > 0.99

i.e. $$P(x \geq 1) > 0.99$$

Therefore, $$1 - P(x = 0) > 0.99$$

or $$1 - {}^nC_0 \frac{1}{4^n} > 0.99$$

or $${}^nC_0 \frac{1}{4^n} < 0.01 \text{ i.e. } \frac{1}{4^n} < 0.01$$

or $$4^n > \frac{1}{0.01} = 100 \qquad \ldots (1)$$

The minimum value of n to satisfy the inequality (1) is 4.

Thus, the shooter must fire 4 times.

Example 36 A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

Solution Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, $$P(S) = \frac{1}{6},\ P(F) = \frac{5}{6}$$

$$P(\text{A wins in the first throw}) = P(S) = \frac{1}{6}$$

A gets the third throw, when the first throw by A and second throw by B result into failures.

Therefore, $$P(\text{A wins in the 3rd throw}) = P(FFS) = P(F)P(F)P(S) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$$

$$= \left(\frac{5}{6}\right)^2 \times \frac{1}{6}$$

$$P(\text{A wins in the 5th throw}) = P(FFFFS) = \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) \text{ and so on.}$$

Hence, $$P(\text{A wins}) = \frac{1}{6} + \left(\frac{5}{6}\right)^2 \left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^4 \left(\frac{1}{6}\right) + \ldots$$

$$= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$$

$$P(\text{B wins}) = 1 - P(\text{A wins}) = 1 - \frac{6}{11} = \frac{5}{11}$$

Remark If $a + ar + ar^2 + ... + ar^{n-1} + ...$, where $|r| < 1$, then sum of this infinite G.P. is given by $\frac{a}{1-r}$. (Refer A.1.3 of Class XI Text book).

Example 37 If a machine is correctly set up, it produces 90% acceptable items. If it is incorrectly set up, it produces only 40% acceptable items. Past experience shows that 80% of the set ups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly setup.

Solution Let A be the event that the machine produces 2 acceptable items.

Also let B_1 represent the event of correct set up and B_2 represent the event of incorrect setup.

Now $P(B_1) = 0.8$, $P(B_2) = 0.2$

$$P(A|B_1) = 0.9 \times 0.9 \text{ and } P(A|B_2) = 0.4 \times 0.4$$

Therefore
$$P(B_1|A) = \frac{P(B_1)\,P(A|B_1)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2)}$$

$$= \frac{0.8 \times 0.9 \times 0.9}{0.8 \times 0.9 \times 0.9 + 0.2 \times 0.4 \times 0.4} = \frac{648}{680} = 0.95$$

Summary

The salient features of the chapter are –

- The conditional probability of an event E, given the occurrence of the event F is given by $P(E|F)=\dfrac{P(E\cap F)}{P(F)}$, $P(F)\neq 0$
- $0\le P(E|F)\le 1$, $P(E'|F)=1-P(E|F)$

  $P((E\cup F)|G)=P(E|G)+P(F|G)-P((E\cap F)|G)$
- $P(E\cap F)=P(E)\,P(F|E)$, $P(E)\neq 0$

  $P(E\cap F)=P(F)\,P(E|F)$, $P(F)\neq 0$
- If E and F are independent, then

  $P(E\cap F)=P(E)\,P(F)$

  $P(E|F)=P(E)$, $P(F)\neq 0$

  $P(F|E)=P(F)$, $P(E)\neq 0$
- **Theorem of total probability**

  Let $\{E_1, E_2, ..., E_n\}$ be a partition of a sample space and suppose that each of $E_1, E_2, ..., E_n$ has nonzero probability. Let A be any event associated with S, then

  $P(A)=P(E_1)\,P(A|E_1)+P(E_2)\,P(A|E_2)+...+P(E_n)\,P(A|E_n)$
- **Bayes' theorem** If $E_1, E_2, ..., E_n$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, ..., E_n$ are pairwise disjoint and $E_1\cup E_2\cup...\cup E_n=S$ and A be any event with nonzero probability, then

$$P(E_i|A)=\frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n}P(E_j)P(A|E_j)}$$

- A random variable is a real valued function whose domain is the sample space of a random experiment.
- The probability distribution of a random variable X is the system of numbers

X	:	x_1	x_2	...	x_n
P(X)	:	p_1	p_2	...	p_n

where, $p_i>0,\ \sum_{i=1}^{n}p_i=1,\ i=1,2,...,n$

- Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots p_n$ respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i p_i$.

  The mean of a random variable X is also called the expectation of X, denoted by E (X).

- Let X be a random variable whose possible values $x_1, x_2, \ldots, x_n$ occur with probabilities $p(x_1), p(x_2), \ldots, p(x_n)$ respectively.

  Let $\mu = E(X)$ be the mean of X. The variance of X, denoted by Var (X) or σ_x^2, is defined as $\sigma_x^2 = \text{Var}(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$

  or equivalently $\sigma_x^2 = E(X - \mu)^2$

  The non-negative number

$$\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)}$$

  is called the standard deviation of the random variable X.

- $\text{Var}(X) = E(X^2) - [E(X)]^2$

- Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :
  (i) There should be a finite number of trials.
  (ii) The trials should be independent.
  (iii) Each trial has exactly two outcomes : success or failure.
  (iv) The probability of success remains the same in each trial.

  For Binomial distribution B (n, p), $P(X = x) = {}^nC_x\, q^{n-x} p^x$, $x = 0, 1, \ldots, n$ $(q = 1 - p)$