Miscellaneous Examples

Example 37 Find $\int \cos 6x \sqrt{1+\sin 6x}\, dx$

Solution Put $t = 1 + \sin 6x$, so that $dt = 6 \cos 6x\, dx$

Therefore $$\int \cos 6x \sqrt{1+\sin 6x}\, dx = \frac{1}{6}\int t^{\frac{1}{2}} dt$$

$$= \frac{1}{6} \times \frac{2}{3} (t)^{\frac{3}{2}} + C = \frac{1}{9}(1+\sin 6x)^{\frac{3}{2}} + C$$

Example 38 Find $\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx$

Solution We have $$\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \int \frac{\left(1-\frac{1}{x^3}\right)^{\frac{1}{4}}}{x^4} dx$$

Put $1 - \frac{1}{x^3} = 1 - x^{-3} = t$, so that $\frac{3}{x^4} dx = dt$

Therefore $$\int \frac{(x^4 - x)^{\frac{1}{4}}}{x^5} dx = \frac{1}{3}\int t^{\frac{1}{4}} dt = \frac{1}{3} \times \frac{4}{5} t^{\frac{5}{4}} + C = \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{\frac{5}{4}} + C$$

Example 39 Find $\int \frac{x^4\, dx}{(x-1)(x^2+1)}$

Solution We have

$$\frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{x^3 - x^2 + x - 1}$$

$$= (x+1) + \frac{1}{(x-1)(x^2+1)} \quad \text{... (1)}$$

Now express $$\frac{1}{(x-1)(x^2+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)} \quad \text{... (2)}$$

So $$1 = A(x^2 + 1) + (Bx + C)(x - 1)$$
$$= (A + B)x^2 + (C - B)x + A - C$$

Equating coefficients on both sides, we get $A + B = 0$, $C - B = 0$ and $A - C = 1$, which give $A = \frac{1}{2}$, $B = C = -\frac{1}{2}$. Substituting values of A, B and C in (2), we get

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)} \quad \text{... (3)}$$

Again, substituting (3) in (1), we have

$$\frac{x^4}{(x-1)(x^2+x+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{1}{2}\frac{x}{(x^2+1)} - \frac{1}{2(x^2+1)}$$

Therefore

$$\int \frac{x^4}{(x-1)(x^2+x+1)} dx = \frac{x^2}{2} + x + \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1} x + C$$

Example 40 Find $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

Solution Let $$I = \int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$$

$$= \int \log(\log x)\, dx + \int \frac{1}{(\log x)^2} dx$$

In the first integral, let us take 1 as the second function. Then integrating it by parts, we get

$$I = x \log(\log x) - \int \frac{1}{x \log x} x\, dx + \int \frac{dx}{(\log x)^2}$$

$$= x \log(\log x) - \int \frac{dx}{\log x} + \int \frac{dx}{(\log x)^2} \quad \text{... (1)}$$

Again, consider $\int \frac{dx}{\log x}$, take 1 as the second function and integrate it by parts, we have $$\int \frac{dx}{\log x} = \left[\frac{x}{\log x} - \int x\left\{-\frac{1}{(\log x)^2}\left(\frac{1}{x}\right)\right\} dx\right] \quad \text{... (2)}$$

Putting (2) in (1), we get

$$I = x\log(\log x) - \frac{x}{\log x} - \int \frac{dx}{(\log x)^2} + \int \frac{dx}{(\log x)^2} = x\log(\log x) - \frac{x}{\log x} + C$$

Example 41 Find $\int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx$

Solution We have

$$I = \int \left[\sqrt{\cot x} + \sqrt{\tan x}\right] dx = \int \sqrt{\tan x}\,(1 + \cot x)\, dx$$

Put $\tan x = t^2$, so that $\sec^2 x\, dx = 2t\, dt$

or $$dx = \frac{2t\, dt}{1 + t^4}$$

Then $$I = \int t\left(1 + \frac{1}{t^2}\right)\frac{2t}{(1+t^4)}\, dt.$$

$$= 2\int \frac{(t^2+1)}{t^4+1}\, dt = 2\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t^2+\frac{1}{t^2}\right)} = 2\int \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2 + 2}$$

Put $t - \frac{1}{t} = y$, so that $\left(1 + \frac{1}{t^2}\right) dt = dy$. Then

$$I = 2\int \frac{dy}{y^2 + \left(\sqrt{2}\right)^2} = \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}} + C = \sqrt{2}\tan^{-1}\frac{\left(t - \frac{1}{t}\right)}{\sqrt{2}} + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + C = \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C$$

Example 42 Find $\int \frac{\sin 2x \cos 2x\, dx}{\sqrt{9 - \cos^4(2x)}}$

Solution Let $I = \int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}}\, dx$

Put $\cos^2(2x) = t$ so that $4\sin 2x \cos 2x\, dx = -dt$

Therefore $$I = -\frac{1}{4}\int \frac{dt}{\sqrt{9 - t^2}} = -\frac{1}{4}\sin^{-1}\left(\frac{t}{3}\right) + C = -\frac{1}{4}\sin^{-1}\left[\frac{1}{3}\cos^2 2x\right] + C$$

Example 43 Evaluate $\int_{-1}^{\frac{3}{2}} |x\sin(\pi x)|\, dx$

Solution Here $f(x) = |x \sin \pi x| = \begin{cases} x\sin \pi x \text{ for } -1 \le x \le 1 \\ -x\sin \pi x \text{ for } 1 \le x \le \frac{3}{2} \end{cases}$

Therefore $$\int_{-1}^{\frac{3}{2}} |x\sin \pi x|\, dx = \int_{-1}^{1} x\sin \pi x\, dx + \int_{1}^{\frac{3}{2}} -x\sin \pi x\, dx$$

$$= \int_{-1}^{1} x\sin \pi x\, dx - \int_{1}^{\frac{3}{2}} x\sin \pi x\, dx$$

Integrating both integrals on righthand side, we get

$$\int_{-1}^{\frac{3}{2}} |x\sin \pi x|\, dx = \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{-x\cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$

$$= \frac{2}{\pi} - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3}{\pi} + \frac{1}{\pi^2}$$

Example 44 Evaluate $\int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2\sin^2 x}$

Solution Let $I = \int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2\sin^2 x} = \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$ (using P_4)

$$= \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} - \int_0^{\pi} \frac{x\, dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} - I$$

Thus $$2I = \pi\int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

or

$$I = \frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{2}\cdot 2\int_0^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} \text{ (using P}_6\text{)}$$

$$= \pi\left[\int_0^{\frac{\pi}{4}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}\right]$$

$$= \pi\left[\int_0^{\frac{\pi}{4}}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\operatorname{cosec}^2 x\,dx}{a^2\cot^2 x + b^2}\right]$$

$$= \pi\left[\int_0^1\frac{dt}{a^2 + b^2t^2} - \int_1^0\frac{du}{a^2u^2 + b^2}\right]\left(put \tan x = t \text{ and } \cot x = u\right)$$

$$= \frac{\pi}{ab}\left[\tan^{-1}\frac{bt}{a}\right]_0^1 - \frac{\pi}{ab}\left[\tan^{-1}\frac{au}{b}\right]_1^0 = \frac{\pi}{ab}\left[\tan^{-1}\frac{b}{a} + \tan^{-1}\frac{a}{b}\right] = \frac{\pi^2}{2ab}$$

Summary

- Integration is the inverse process of differentiation. In the differential calculus we are given a function and we have to find the derivative or differential of this function, but in the integral calculus, we are to find a function whose differential is given. Thus, integration is a process which is the inverse of differentiation.

  Let $\frac{d}{dx}F(x) = f(x)$. Then we write $\int f(x)\,dx = F(x) + C$. These integrals are called indefinite integrals or general integrals, C is called constant of integration. All these integrals differ by a constant.

- From the geometric point of view, an indefinite integral is collection of family of curves, each of which is obtained by translating one of the curves parallel to itself upwards or downwards along the y-axis.
- Some properties of indefinite integrals are as follows:

  1. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

  2. For any real number k, $\int k\,f(x)\,dx = k\int f(x)\,dx$

  More generally, if $f_1, f_2, f_3, \ldots, f_n$ are functions and $k_1, k_2, \ldots, k_n$ are real numbers. Then

$$\int [k_1 f_1(x) + k_2 f_2(x) + \ldots + k_n f_n(x)]\,dx$$

$$= k_1 \int f_1(x)\,dx + k_2 \int f_2(x)\,dx + \ldots + k_n \int f_n(x)\,dx$$

Some standard integrals

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$. Particularly, $\int dx = x + C$

(ii) $\int \cos x \, dx = \sin x + C$

(iii) $\int \sin x \, dx = -\cos x + C$

(iv) $\int \sec^2 x \, dx = \tan x + C$

(v) $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

(vi) $\int \sec x \tan x \, dx = \sec x + C$

(vii) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(viii) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

(ix) $\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + C$

(x) $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

(xi) $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

(xii) $\int e^x dx = e^x + C$

(xiii) $\int a^x dx = \frac{a^x}{\log a} + C$

(xiv) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$

(xv) $\int \frac{dx}{x\sqrt{x^2-1}} = -\operatorname{cosec}^{-1} x + C$

(xvi) $\int \frac{1}{x} dx = \log |x| + C$

Integration by partial fractions

Recall that a rational function is ratio of two polynomials of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. If degree of the polynomial $P(x)$ is greater than the degree of the polynomial $Q(x)$, then we may divide $P(x)$ by $Q(x)$ so that $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where $T(x)$ is a polynomial in x and degree of $P_1(x)$ is less than the degree of $Q(x)$. $T(x)$ being polynomial can be easily integrated. $\frac{P_1(x)}{Q(x)}$ can be integrated by expressing $\frac{P(x)}{Q(x)}$ as the sum of partial fractions of the following type:

1. $$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a}+\frac{B}{x-b},\ a \neq b$$

2. $$\frac{px+q}{(x-a)^2} = \frac{A}{x-a}+\frac{B}{(x-a)^2}$$

3. $$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$$

4. $$\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$$

5. $$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$$

where x^2+bx+c can not be factorised further.

- **Integration by substitution**

A change in the variable of integration often reduces an integral to one of the fundamental integrals. The method in which we change the variable to some other variable is called the method of substitution. When the integrand involves some trigonometric functions, we use some well known identities to find the integrals. Using substitution technique, we obtain the following standard integrals.

(i) $\int \tan x\, dx = \log|\sec x| + C$ (ii) $\int \cot x\, dx = \log|\sin x| + C$

(iii) $\int \sec x\, dx = \log|\sec x + \tan x| + C$

(iv) $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$

- **Integrals of some special functions**

(i) $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(ii) $$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$ (iii) $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(iv) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+C$ (v) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}+C$

(vi) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|+C$

- **Integration by parts**

For given functions f_1 and f_2, we have

$\int f_1(x)\cdot f_2(x)\,dx = f_1(x)\int f_2(x)\,dx - \int\left[\frac{d}{dx}f_1(x)\cdot\int f_2(x)\,dx\right]dx$, i.e., the integral of the product of two functions = first function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

- $\int e^x[f(x)+f'(x)]\,dx = \int e^x f(x)\,dx + C$

- **Some special types of integrals**

(i) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

(ii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

(iii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

(iv) Integrals of the types $\int \frac{dx}{ax^2+bx+c}$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ can be transformed into standard form by expressing

$$ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

(v) Integrals of the types $\int \frac{px+q\,dx}{ax^2+bx+c}$ or $\int \frac{px+q\,dx}{\sqrt{ax^2+bx+c}}$ can be transformed into standard form by expressing

$px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A(2ax + b) + B$, where A and B are determined by comparing coefficients on both sides.

- We have defined $\int_a^b f(x)\,dx$ as the area of the region bounded by the curve $y = f(x)$, $a \le x \le b$, the x-axis and the ordinates $x = a$ and $x = b$. Let x be a given point in $[a, b]$. Then $\int_a^x f(x)\,dx$ represents the **Area function** A (x). This concept of area function leads to the Fundamental Theorems of Integral Calculus.

- **First fundamental theorem of integral calculus**

  Let the area function be defined by $A(x) = \int_a^x f(x)\,dx$ for all $x \ge a$, where the function f is assumed to be continuous on $[a, b]$. Then $A'(x) = f(x)$ for all $x \in [a, b]$.

- **Second fundamental theorem of integral calculus**

  Let f be a continuous function of x defined on the closed interval $[a, b]$ and let F be another function such that $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f, then $\int_a^b f(x)\,dx = \left[F(x) + C\right]_a^b = F(b) - F(a)$.

  This is called the definite integral of f over the range $[a, b]$, where a and b are called the limits of integration, a being the lower limit and b the upper limit.